Nested Padé approximants as a tool for multivariate finite element problems with porous media

SAPEM 2014, Stockholm, December 16, 2014

Romain Rumpler   – CREO Dynamics AB, Linköping
Peter Göransson   – KTH, Stockholm
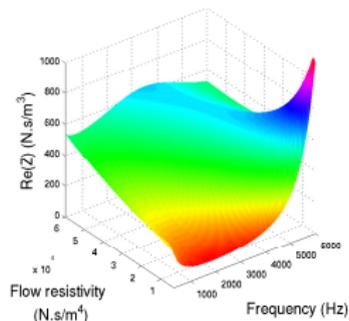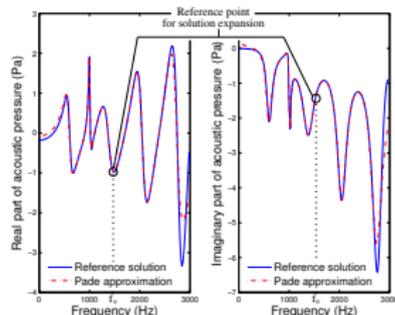Henry Rice   – Trinity College Dublin

sapem 2014

Motivations and objectives

- Establish a Padé-based approach for fast parametric sweeps of general problems

- Extend the method to multivariate problems

- Account for non trivial parametric analyses

- Potential applications:

    ⋆ Optimization problems

    ⋆ Response for stochastic systems

    ⋆ ...

Motivations   An extension to multivariate expansions?   Bivariate Padé reconstruction   Conclusion

Some approaches from the literature

Motivations

An extension to multivariate expansions?
Some approaches from the literature
The nested Padé approximants
Test case: acoustic absorption

Bivariate Padé reconstruction

Conclusion

A few approaches of potential interest...

- Homogeneous multivariate Padé approximants [Cuyt, 1983,1985,1997]

    ✓ Efficient recursive algorithms developped

    ✓ Convergence proofs

    ✗ Rapid increase of numerator and denominator polynomial degrees

    ✗ Singularities potentially appearing in the convergence domain

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    ✗ Recursive increase of the polynomial degrees of the multivariate approximant

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- Nested Padé approximants [Guillaume, 1997]

  ✓ Univariate Padé approximants, convergence proofs and straightforward implementation

  ✓ As many final coefficients as the initial multivariate Taylor expansion

  ✗ Performance convergence/cost?

The nested Padé approximants: main steps, bivariate case (1/3)

- General form of the problems to solve: $\mathbf{Z}(x, y)\mathbf{u}(x, y) = \mathbf{f}(x, y)$

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  ⋆ $s_i^{N_j}(\Delta y) = \sum_{j=0}^{N_j} s_{ij}\Delta y^j \Longrightarrow s_{ij}? \Longrightarrow \begin{cases} \mathbf{H}_0\mathbf{S}_0 = \mathbf{C}_0 \\ \\ \mathbf{H}_0\mathbf{S}_j = \mathbf{C}_j - \displaystyle\sum_{k=1}^{j} \mathbf{H}_k\mathbf{S}_{j-k}, \ \ j = 1 \cdots N_j \end{cases}$

The nested Padé approximants: main steps, bivariate case (2/3)

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  ⋆ $s_i^{N_j}(\Delta y)$ with Padé $\Longrightarrow s_i^{N_j}(\Delta y) = \dfrac{\sum_{j=0}^{m_j} t_{ij}\Delta y^j}{1 + \sum_{j=1}^{n_j} t_{i(m_j+j)}\Delta y^j}$, with $m_j + n_j = N_j$

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  May be rewritten as a system of linear equations:

  $$[\mathbf{A_i}]\begin{bmatrix} t_{i(m_j+1)} \\ \vdots \\ t_{i(N_j)} \\ t_{i0} \\ \vdots \\ t_{i(m_j)} \end{bmatrix} = \begin{bmatrix} s_{i0} \\ \vdots \\ s_{i(n_j-1)} \\ s_{i(n_j)} \\ \vdots \\ s_{i(N_j)} \end{bmatrix}, \text{ for } i = 1, \ldots, N_i + 1$$

The nested Padé approximants: main steps, bivariate case (3/3)

- Nested Padé solution expansion around (x_0, y_0), for $\mathbf{Z}(x_0, y_0)\mathbf{u}(x_0, y_0) = \mathbf{f}(x_0, y_0)$?

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      $\left[(N_i + 1) * (N_j + 1) - 1\right]$ multiple RHS full solutions

The nested Padé approximants: main steps, bivariate case (3/3)

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  ▶ **Rebuild solution** around (x_0, y_0): $\mathbf{u}(x_0 + \Delta x, y_0 + \Delta y)$ (simple algebraic evaluation)

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Test case: acoustic absorption

Motivations

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# Test case: equivalent fluid porous material

- Absorption problem:

[Y. Miki, *Acoustical properties of porous materials: Modifications of delany-bazley models*, JASA, 1990]

Test case: equivalent fluid porous material

- Absorption problem:

- **Delany-Bazley-Miki empirical model**, equivalent fluid (2.5 cm-thick layer)
- Dependence on **two parameters**: frequency f and flow resistivity σ
  $f : [300 - 6000]$Hz and $\sigma : [3000 - 60000]$N.s/m^4
- **Density** and **bulk modulus**: $\widetilde{\rho}_{eq}(f, \sigma)$, $\widetilde{K}_{eq}(f, \sigma)$
- Form of the problem to be solved: $\mathbf{Q_p}(f, \sigma) \mathbf{P}(f, \sigma) = \mathbf{V}$

[Y. Miki, *Acoustical properties of porous materials: Modifications of delany-bazley models*, JASA, 1990]

Acoustic impedance (Real part), $(f_0, \sigma_0) = (2500\ \text{Hz}, 25000\ \text{N.s/m}^4)$

- Reference solution

- 1-point bivariate reconstruction

Test case: acoustic absorption
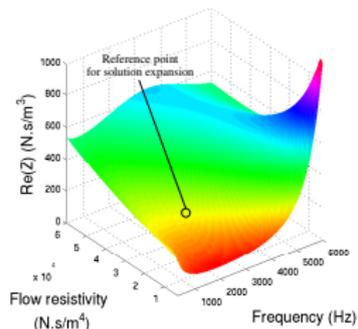
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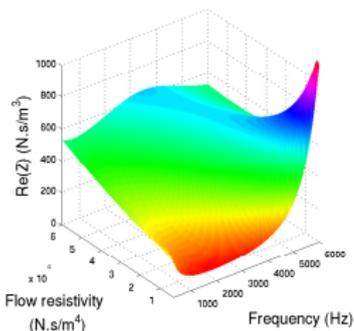
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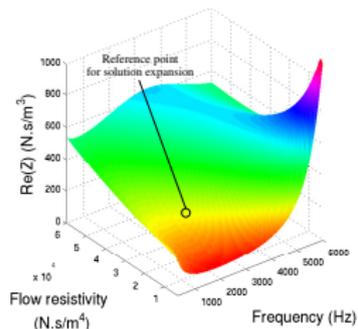
- Relative error ($N_f = N_\sigma = 9$)
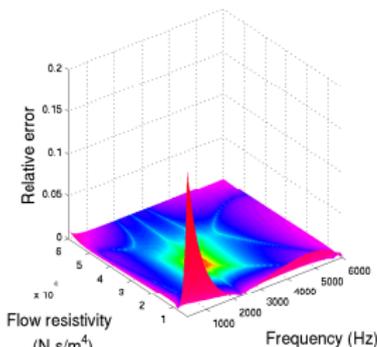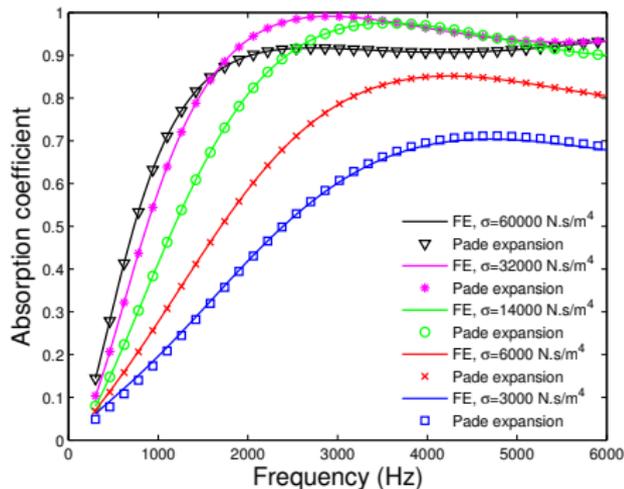
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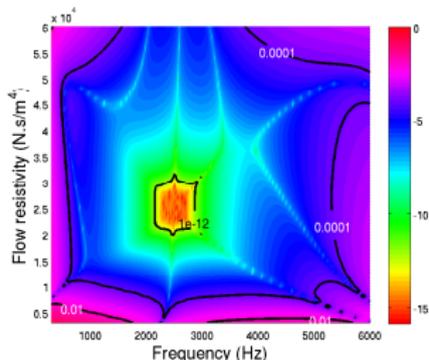
Absorption coefficient, $(f_0, \sigma_0) = (2500 \text{ Hz}, 25000 \text{ N.s/m}^4)$

- Reference/reconstructed solution
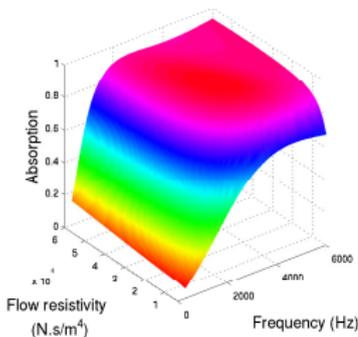
- Relative error ($N_f = N_\sigma = 9$)

- Absorption curves
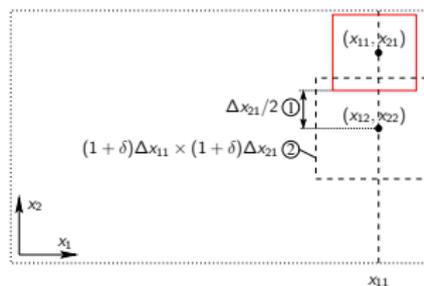
Reconstruction steps over multiple intervals
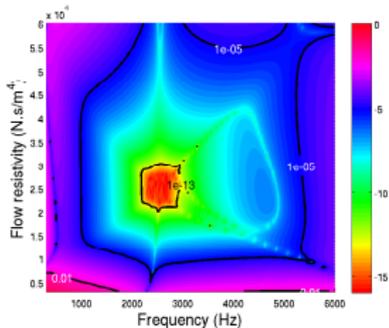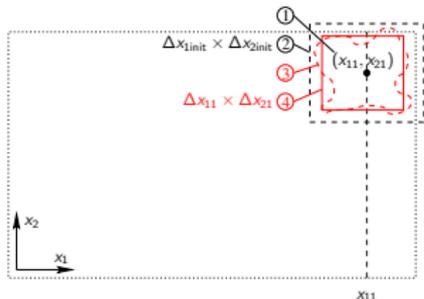
Steps 1–3

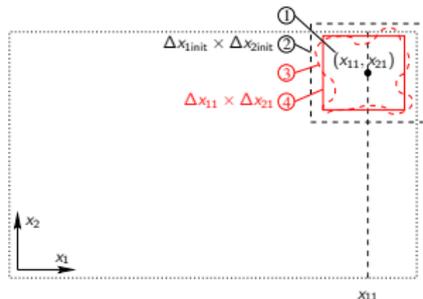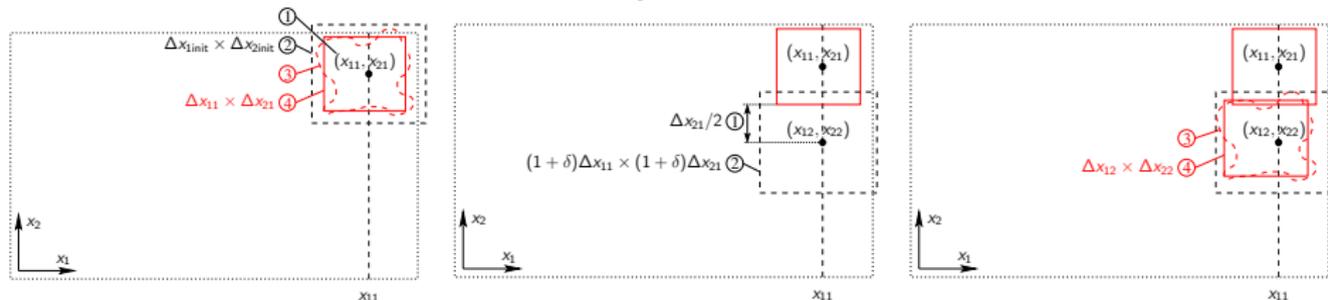# Reconstruction steps over multiple intervals

Steps 1–3

Reconstruction steps over multiple intervals

Steps 1–3

Subsequent steps & further details:

[**Rumpler R., Göransson P., Deü J.-F.**], *A finite element approach combining a reduced-order system, Padé approximants, and an adaptive frequency windowing for fast multi-frequency solution of poro-acoustic problems*, IJNME, 97(10):1097-0207, 2014.

[**Rumpler R., Göransson P., Rice H.J.**], *An adaptive strategy for the bivariate solution of finite element problems using multivariate nested Padé approximants*, IJNME, 100:689–710, 2014.

Motivations   An extension to multivariate expansions?   Bivariate Padé reconstruction   Conclusion
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3D elasto-poro-acoustic problem

Motivations An extension to multivariate expansions? Bivariate Padé reconstruction Conclusion
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3D elasto-poro-acoustic problem

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- Absorption problem:

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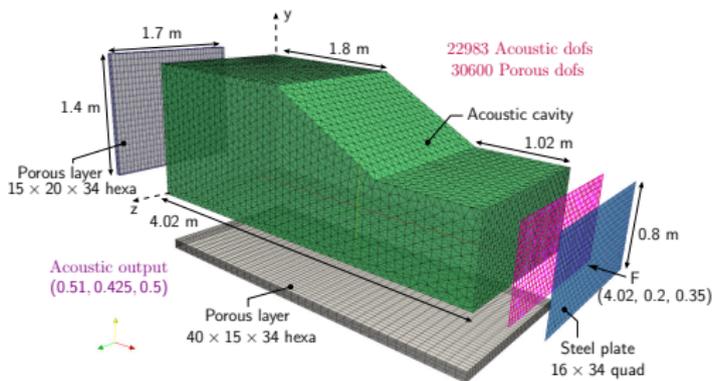3D elasto-poro-acoustic problem
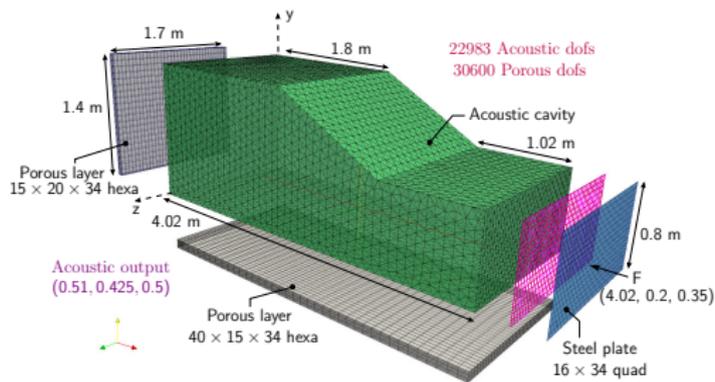
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$$
\begin{bmatrix}
\mathbf{K_s} - 4\pi^2 f^2 \mathbf{M_s} & -\mathbf{H_{sa}} \\
-\mathbf{H_{sa}}^T & \frac{1}{4\pi^2 f^2 \rho_0}\mathbf{K_a} + \frac{1}{4\pi^2 f^2 \widetilde{\rho}_{eq}(f,\sigma)}\mathbf{K_p} - \\
& \frac{1}{K_0}\mathbf{M_a} - \frac{1}{\widetilde{K}_{eq}(f,\sigma)}\mathbf{M_p}
\end{bmatrix}
\begin{bmatrix}
\mathbf{U}(f, \sigma) \\
\mathbf{P}(f, \sigma)
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{F} \\
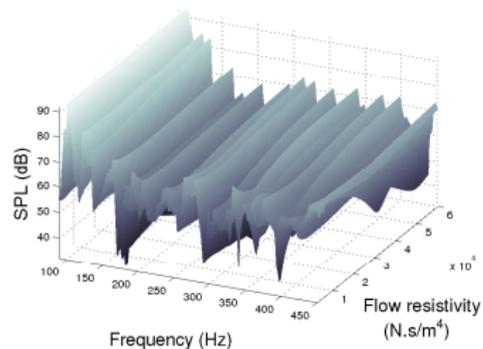\mathbf{0}
\end{bmatrix}
$$

Sound pressure level surface response
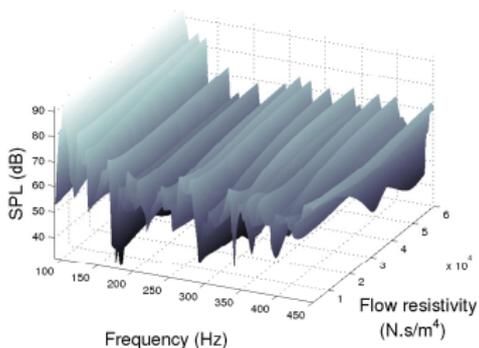
- Reference solution
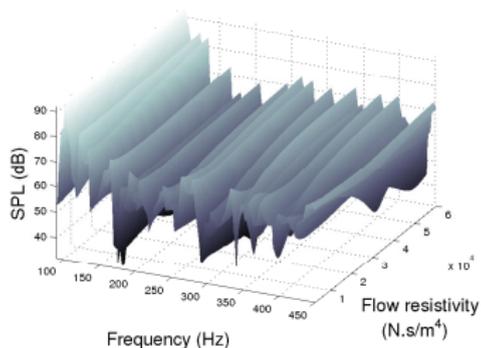
- Padé bivariate reconstruction

Sound pressure level surface response
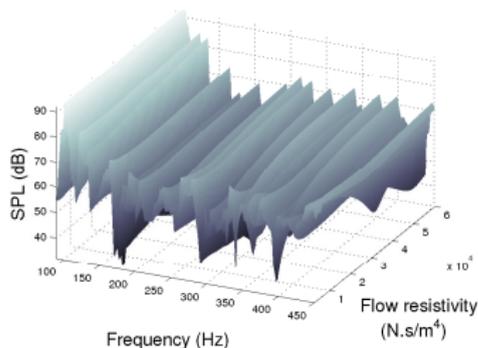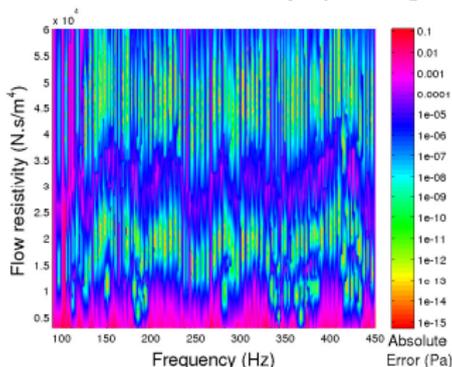
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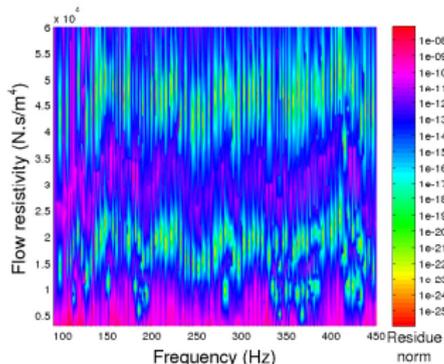- Padé bivariate reconstruction

- Relative error ($N_f = N_\sigma = 7$)
- Residue norm ($N_f = N_\sigma = 7$)

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3D elasto-poro-acoustic problem

CPU time comparison – Padé reconstruction

- CPU reference vs. Nested Padé
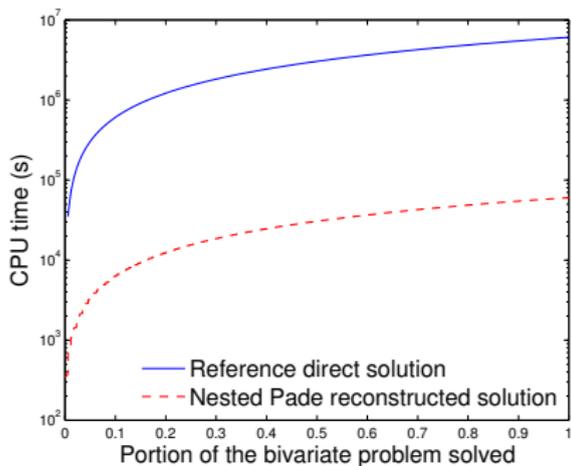
- Speedup factor
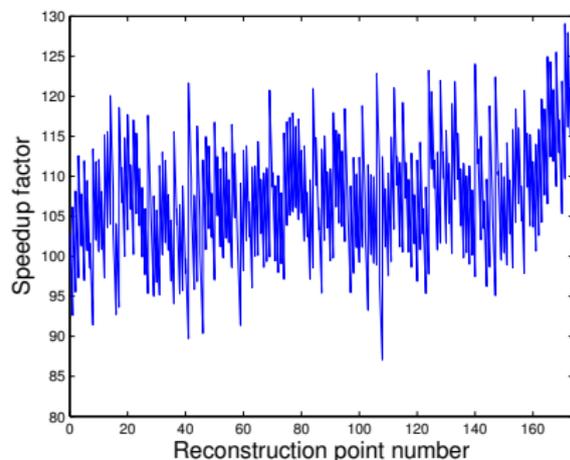
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Main conclusions

- **Good potential** using Padé-based interpolatory approaches for univariate and multivariate problems:
  - ▶ Good convergence properties on univariate and bivariate test cases
  - ▶ Based on univariate Padé approximants
  - ▶ Suited for an efficient implementation
  - ▶ Particularly interesting for partial field reconstruction

- Provides a piece-wise analytical approximate solution

- A numerical tool with several potential engineering applications

- An adaptive reconstruction – no a priori knowledge of the solution required

Conclusion

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Main conclusions

- **Good potential** using Padé-based interpolatory approaches for univariate and multivariate problems:
  - ▶ Good convergence properties on univariate and bivariate test cases
  - ▶ Based on univariate Padé approximants
  - ▶ Suited for an efficient implementation
  - ▶ Particularly interesting for partial field reconstruction

- Provides a piece-wise analytical approximate solution

- A numerical tool with several potential engineering applications

- An adaptive reconstruction – no a priori knowledge of the solution required

THANK YOU FOR YOUR ATTENTION!

Control of the Padé approximation error

- Padé-approximate solution of the general system
  - $\star$ $\mathbf{K}(x_1, x_2)\hat{\mathbf{U}}(x_1, x_2) = \mathbf{F}(x_1, x_2)$

- Partially reconstructed Padé solution:
  - $\star$ $\hat{\mathbf{U}} = \mathbf{\Phi}\hat{\mathbf{U}}_\Phi$

- Residue associated with the partially reconstructed solution at any (x_1, x_2):
  - $\star$ $\mathbf{R}_\alpha(x_1, x_2) = \mathbf{\Phi}^\mathsf{T}\mathbf{K}(x_1, x_2)\mathbf{\Phi}\hat{\mathbf{U}}_\Phi(x_1, x_2) - \mathbf{\Phi}^\mathsf{T}\mathbf{F}(x_1, x_2)$.

- Padé approximation error estimation around a given $(x_{1\alpha}, x_{2\alpha})$:
  - $\star$ $\varepsilon_\alpha(x_1, x_2) = \dfrac{\|\mathbf{R}_\alpha(x_1, x_2)\|_2}{R_{\min}(x_{1\alpha}, x_{2\alpha})}$

  With $R_{\min}(x_{1\alpha}, x_{2\alpha}) = \min_{(x_1, x_2) \neq (x_{1\alpha}, x_{2\alpha})} \|\mathbf{R}_\alpha(x_1, x_2)\|_2$